Snake.

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Joint work with Pascal Bianchi and Walid Hachem

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Stochastic Gradient algorithm

General Problem:

$$\min_{x \in \mathcal{X}} F(x)$$

with F convex over $\mathcal{X}$, Euclidean space.
If F differentiable, Gradient algorithm:

$$x_{n+1} = x_n - \gamma \nabla F(x_n)$$

In ML, ∇F often intractable. Stochastic Gradient algorithm:

$$x_{n+1} = x_n - \gamma_n \nabla_x f(x_n, \xi_{n+1})$$

with (ξ_n) iid, and

$$\mathbb{E}_\xi(f(x, \xi_1)) = F(x).$$

Proximal Gradient algorithm

General Problem:

$$\min_{x \in \mathcal{X}} F(x) + R(x)$$

with F, R convex over $\mathcal{X}$, Euclidean space.

If F differentiable, Proximal Gradient algorithm:

$$x_{n+1} = \mathrm{prox}_{\gamma R}(x_n - \gamma \nabla F(x_n))$$

where the **proximity operator**

$$\mathrm{prox}_{\gamma R}(x) = \arg\min_{y \in \mathcal{X}} \frac{1}{2\gamma}\|x - y\|^2 + R(y).$$

Stochastic Proximal Gradient algorithm

In ML, $\operatorname{prox}_{\gamma R}$ often intractable.
Stochastic Proximal Gradient algorithm:
[Atchadé *et al.*'16],[BH'16]

$$x_{n+1} = \operatorname{prox}_{\gamma_n r(\cdot,\xi_{n+1})}(x_n - \gamma_n \nabla_x f(x_n, \xi_{n+1}))$$

with

- (ξ_n) iid
- $\gamma_n > 0$, $\gamma_n \downarrow 0$
- $\mathbb{E}(f(x, \xi_1)) = F(x)$
- $\mathbb{E}(r(x, \xi_1)) = R(x)$.

Theorem [BH'16] : Under mild assumptions, $x_n \longrightarrow_{n\to+\infty} x_\star$ where $x_\star \in \arg\min_{\mathcal{X}} F + R$ a.s.

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Problem Statement

Consider

- An undirected graph $G = (V, E)$
- A vector of parameters over the nodes $x \in \mathbb{R}^V$
- The **Total Variation** (TV) regularization over G

$$\mathrm{TV}(x, G) = \sum_{\{i,j\} \in E} |x(i) - x(j)|.$$

Our problem:

$$\min_{x \in \mathbb{R}^V} F(x) + \mathrm{TV}(x, G) \tag{1}$$

with $F : \mathbb{R}^V \to \mathbb{R}$ convex, differentiable.

Example: Trend Filtering on Graphs [Wang *et al.*'16]

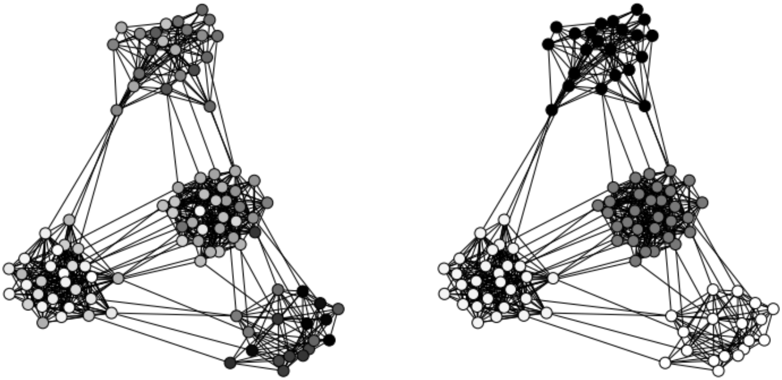


Figure 1: $\min_{x \in \mathbb{R}^V} \frac{1}{2}\|x - y\|^2 + \mathrm{TV}(x, G)$

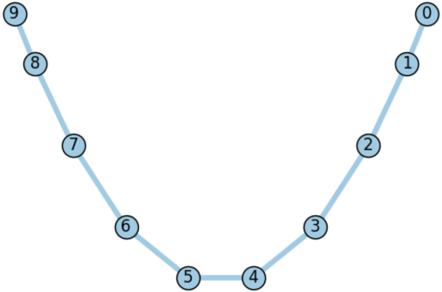
Problem Statement

Proximal Gradient algorithm

$$x_{n+1} = \mathrm{prox}_{\gamma \mathrm{TV}(.,G)}(x_n - \gamma \nabla F(x_n))$$

The computation of $\mathrm{prox}_{\mathrm{TV}(.,G)}(y)$ is

- Fast when the graph G is a path graph : **Taut String algorithm** [Condat'13],[Johnson'13],[Barbero and Sra'14].



- Difficult over general large graphs

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Sampling Random Walks

Let $L \geq 1$.

Let ξ is a stationary simple random walk over G with length $L+1$

$$\mathbb{E}\left(\mathrm{TV}(x,\xi)\right) = \frac{|E|}{L}\mathrm{TV}(x,G).$$

Our problem is equivalent to

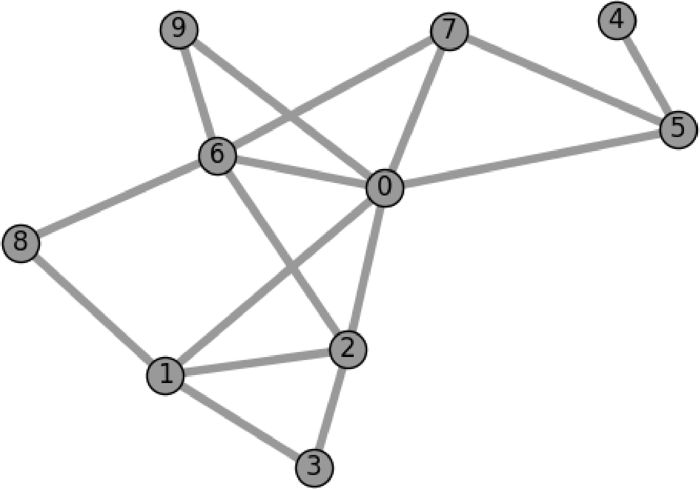
$$\min_{x\in\mathbb{R}^V} LF(x) + |E|\mathbb{E}\left(\mathrm{TV}(x,\xi)\right).$$

Stochastic Proximal Gradient algorithm:

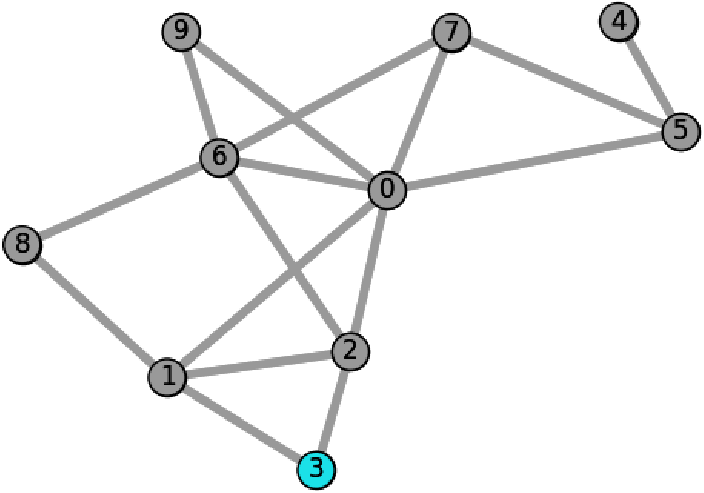
$$\begin{cases} \text{Sample the Stationary Random Walk } \xi_{n+1} \text{ with length } L+1 \\ x_{n+1} = \mathrm{prox}_{\gamma_n |E| \mathrm{TV}(\cdot,\xi_{n+1})}(x_n - \gamma_n L \nabla F(x_n)) \end{cases}$$

- (ξ_n) iid
- $\gamma_n > 0$, $\gamma_n \downarrow 0$.

Example : The Graph G



Example : Sampling the Random Walk ξ_{n+1}



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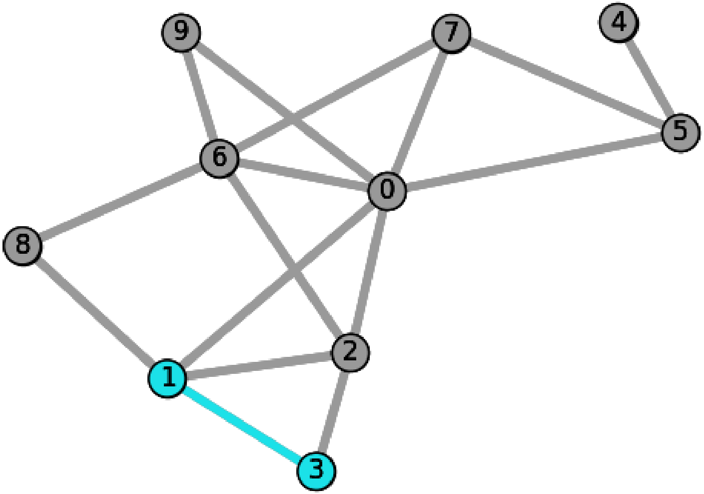

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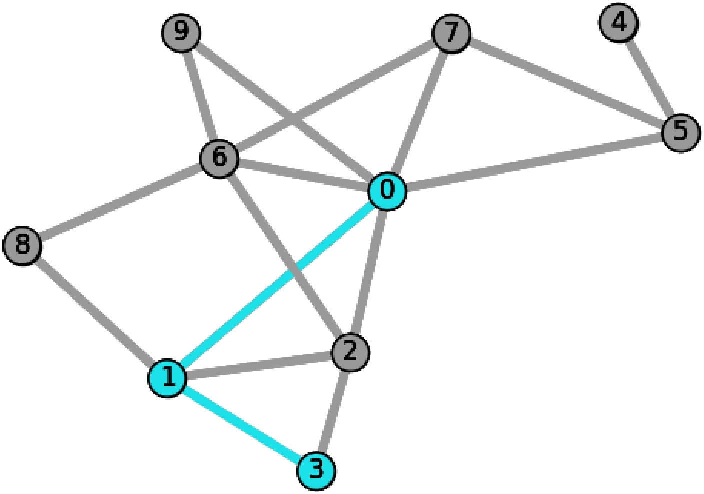

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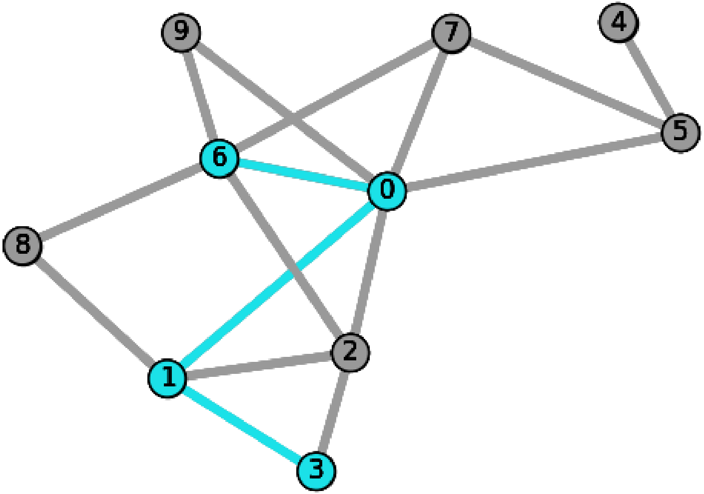

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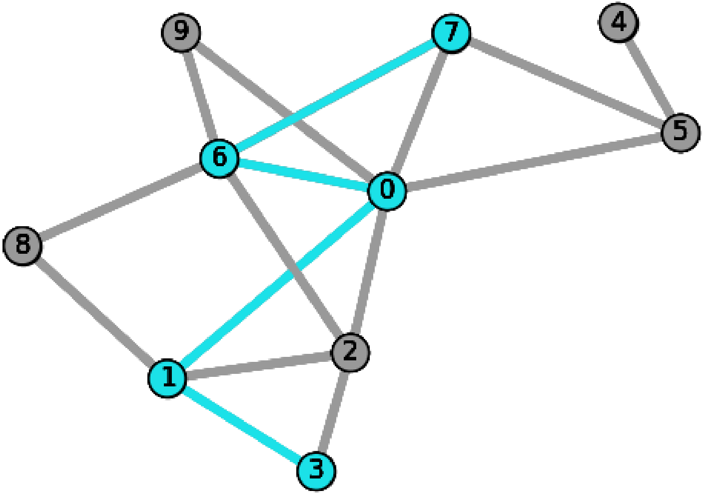
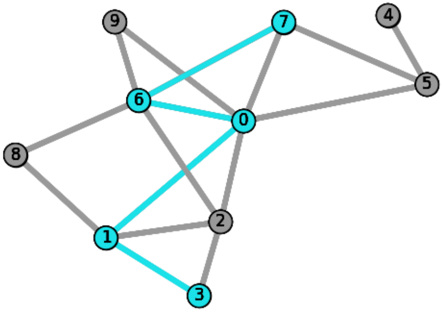


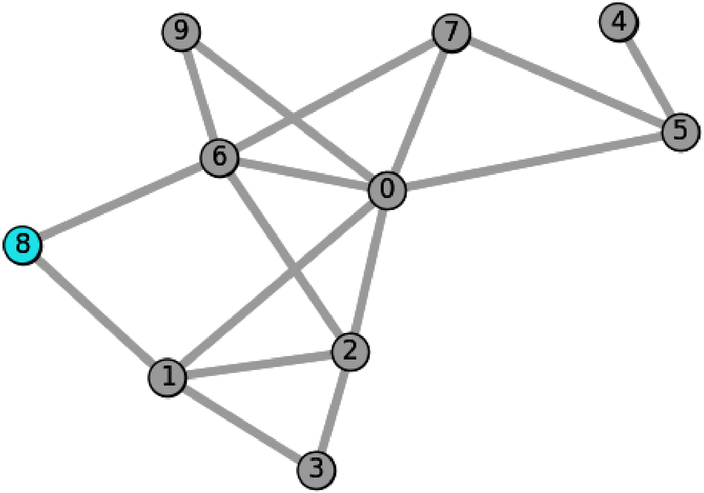
Example : Stochastic Proximal Gradient step



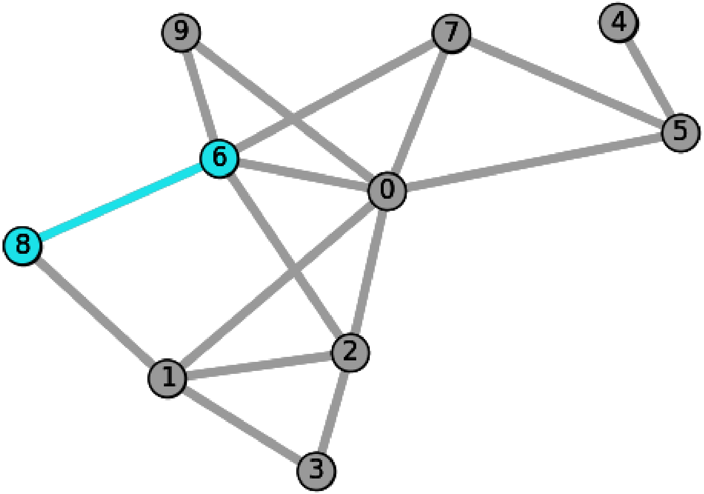
$$\mathrm{TV}(x,\xi_{n+1}) = |x(3){-}x(1)|{+}|x(1){-}x(0)|{+}|x(0){-}x(6)|{+}|x(6){-}x(7)|$$

$$x_{n+1} = \mathrm{prox}_{\gamma_n|E|\mathrm{TV}(\cdot,\xi_{n+1})}(x_n - \gamma_n L\nabla F(x_n))$$

Example : Sampling the Random Walk ξ_{n+2}



Example : Sampling the Random Walk ξ_{n+2}



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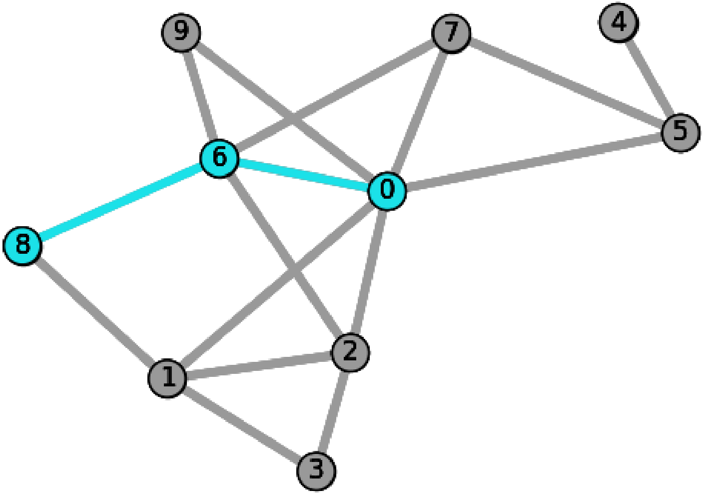

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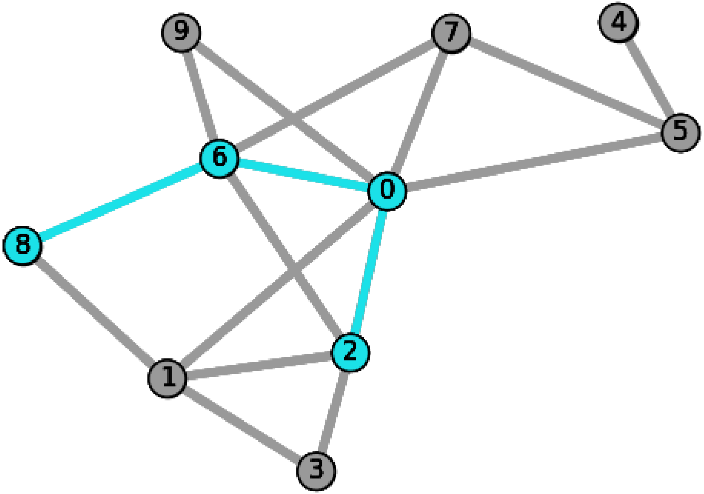
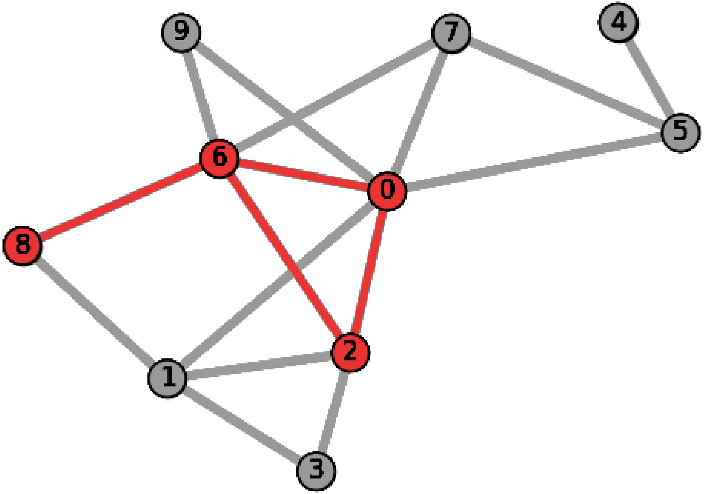
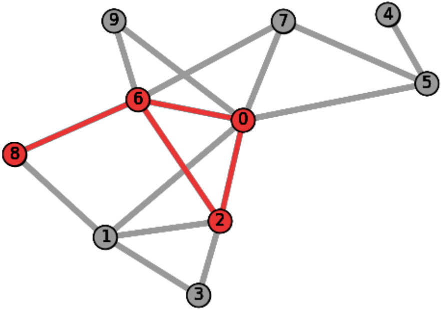


Example : Loop



Example : Stochastic Proximal Gradient step



$$\mathrm{TV}(x, \xi_{n+2}) = |x(8)-x(6)|+|x(6)-x(0)|+|x(0)-x(2)|+|x(2)-x(6)|$$

$$x_{n+2} = \mathrm{prox}_{\gamma_{n+1}|E|\mathrm{TV}(\cdot,\xi_{n+2})}(x_{n+1} - \gamma_{n+1} L \nabla F(x_{n+1}))$$

Problem : ξ_{n+2} is not a path graph

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Snake algorithm

Let ξ is a stationary simple random walk over G with length $L+1$

$$\mathbb{E}\left(\mathrm{TV}(x,\xi)\right) = \frac{|E|}{L}\mathrm{TV}(x,G).$$

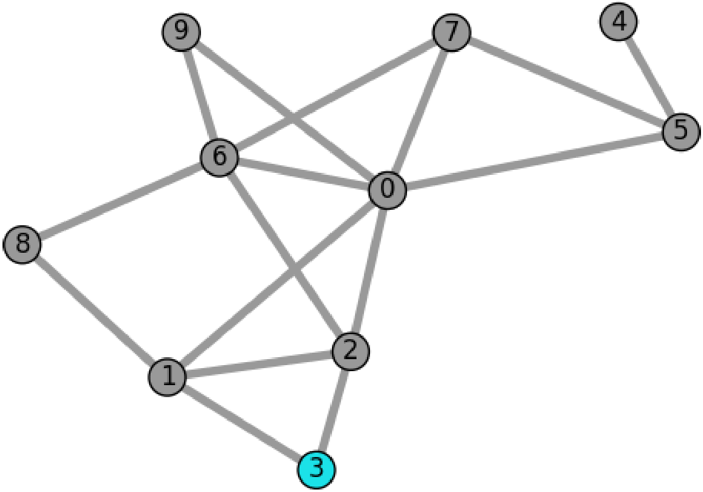
Our problem is equivalent to

$$\min_{x\in\mathbb{R}^V} LF(x) + |E|\mathbb{E}\left(\mathrm{TV}(x,\xi)\right).$$

Snake algorithm:
While unhappy

$$\begin{cases} \text{Sample the Stationary Random Walk } \xi_{n+1} \textbf{ until Loop} \\ x_{n+1} = \mathrm{prox}_{\gamma_n |E| \mathrm{TV}(\cdot,\xi_{n+1})}(x_n - \gamma_n L(\xi_{n+1})\nabla F(x_n)) \end{cases}$$

Example : Snake



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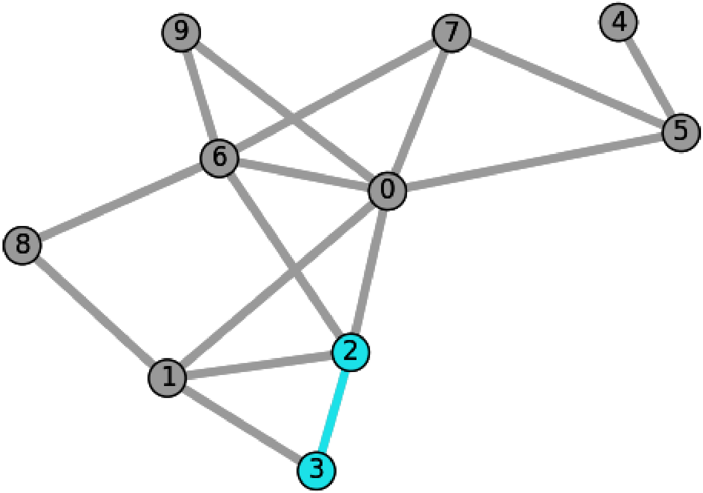

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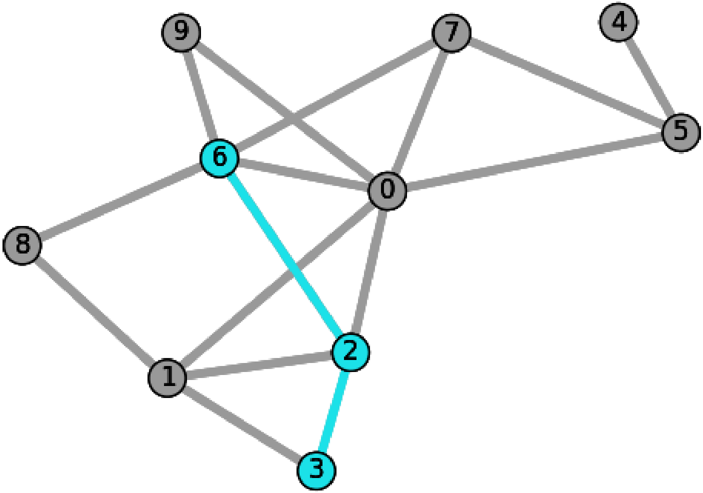

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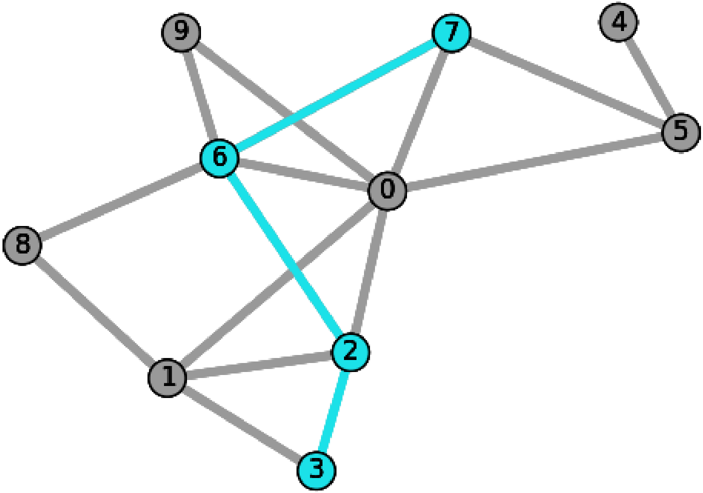

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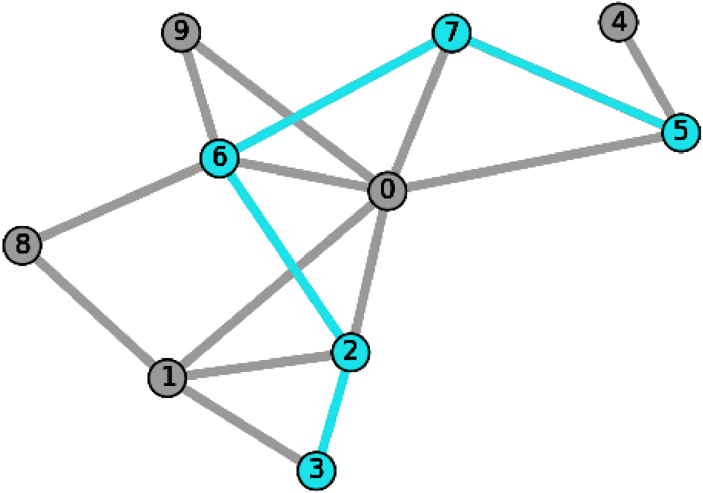

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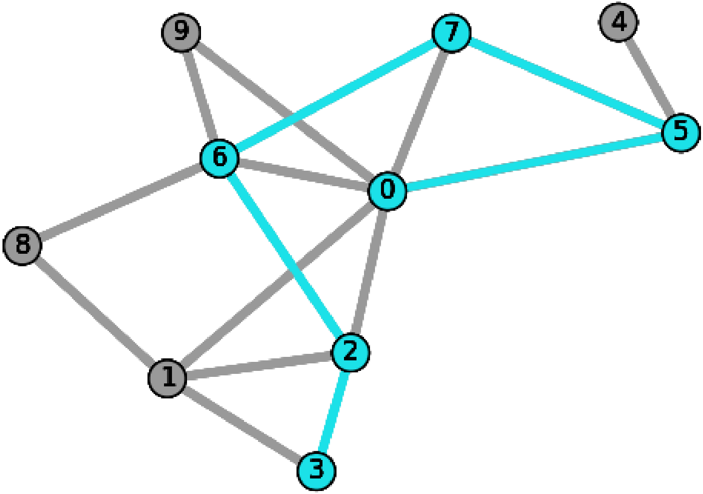

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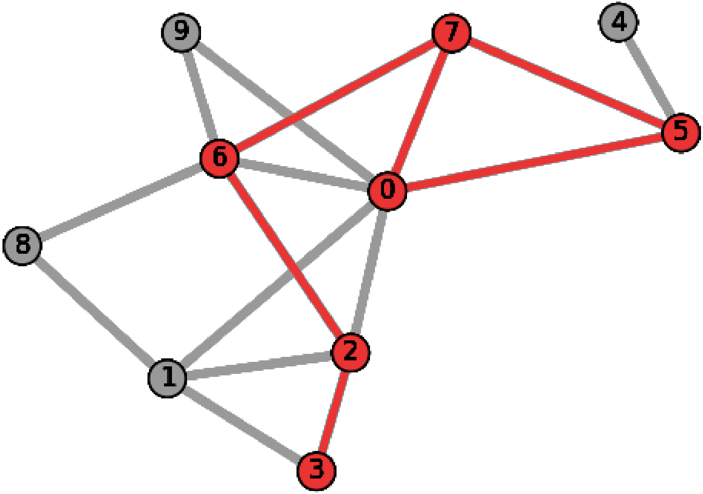

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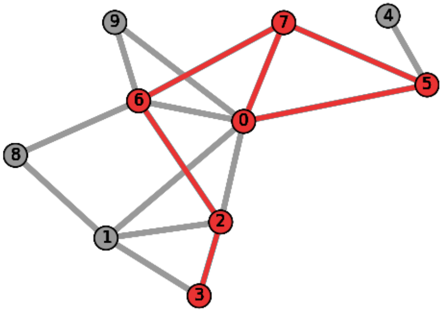
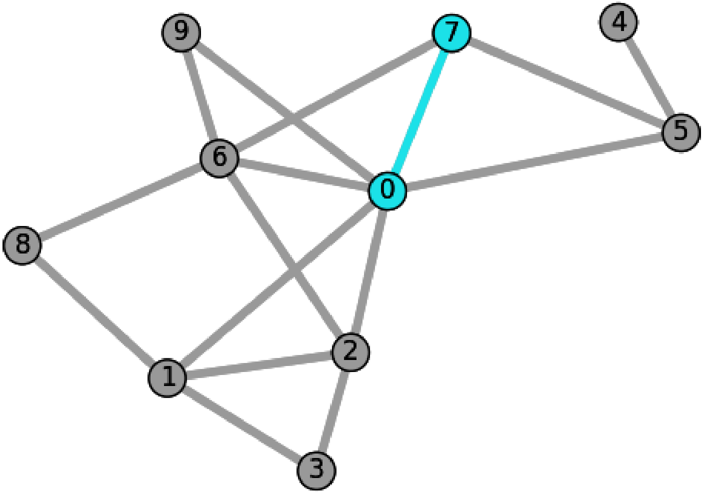


$$\mathrm{TV}(x, \xi_{n+1}) = |x(3) - x(2)| + |x(2) - x(6)| \\ + |x(6) - x(7)| + |x(7) - x(5)| + |x(5) - x(0)|$$

$$x_{n+1} = \mathrm{prox}_{\gamma_n |E| \mathrm{TV}(\cdot, \xi_{n+1})}(x_n - \gamma_n L(\xi_{n+1}) \nabla F(x_n))$$

Example : Snake



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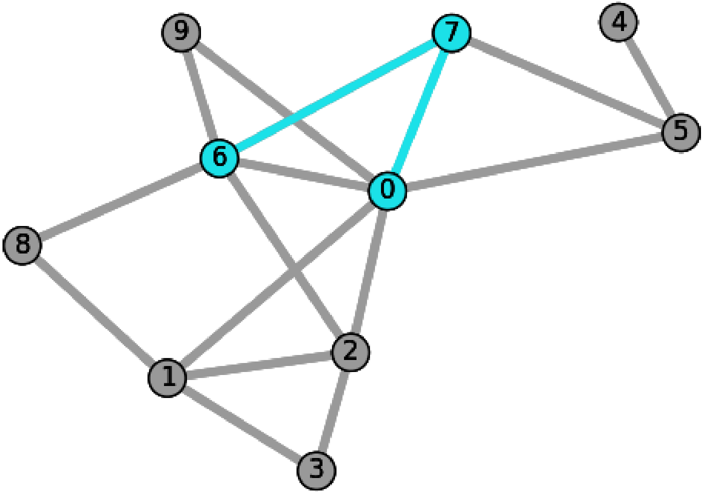

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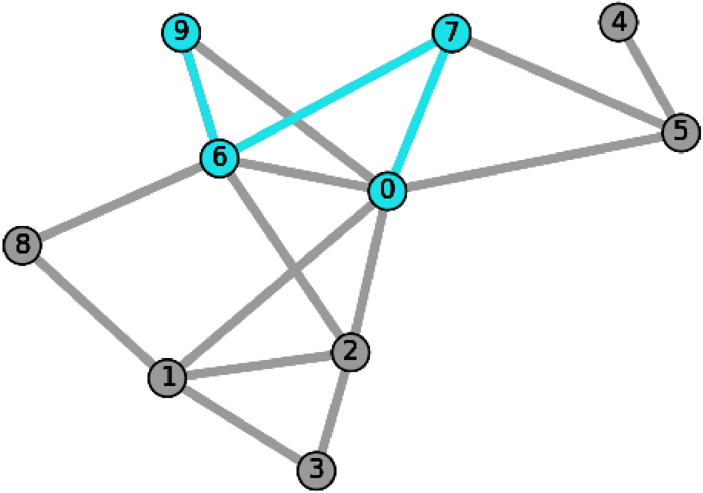

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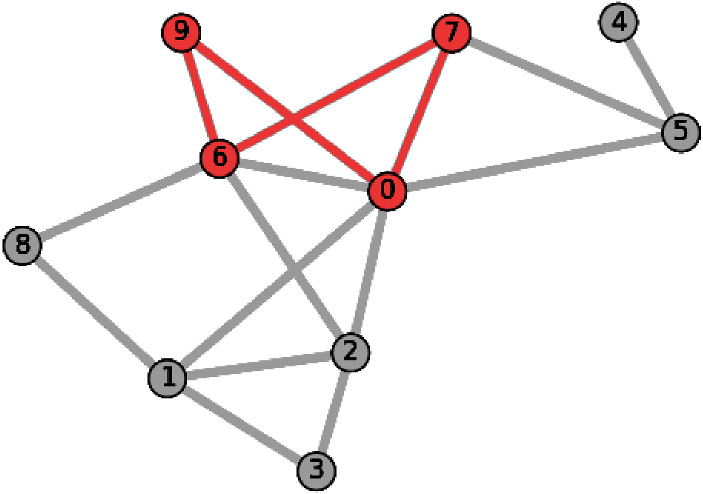

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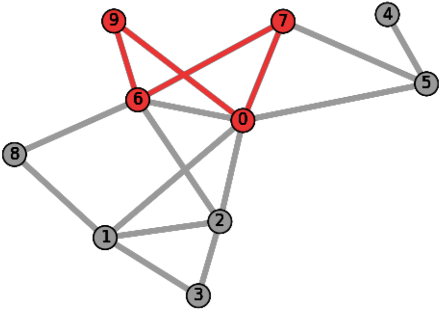
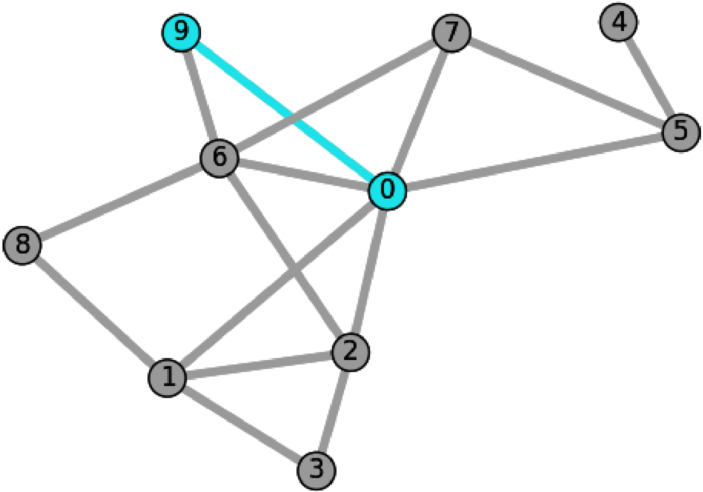


$$\mathrm{TV}(x, \xi_{n+2}) = |x(0) - x(7)| + |x(7) - x(6)| + |x(6) - x(9)|$$

$$x_{n+2} = \mathrm{prox}_{\gamma_{n+1}|E|\mathrm{TV}(\cdot, \xi_{n+2})}(x_{n+1} - \gamma_{n+1} L(\xi_{n+2}) \nabla F(x_{n+1}))$$

Example : Snake



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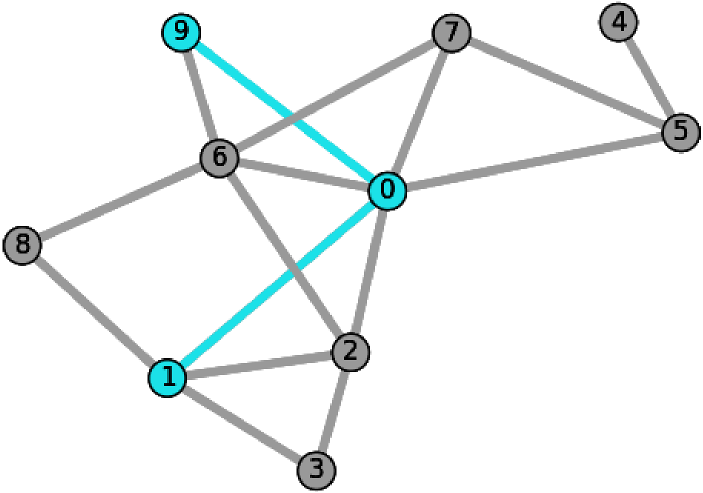

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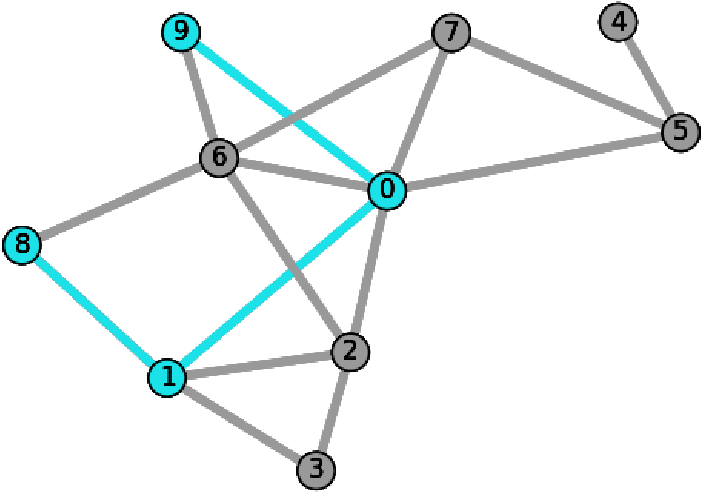

Convergence of Snake algorithm

Snake algorithm:
While unhappy

$$\begin{cases} \text{Sample the Stationary Random Walk } \xi_{n+1} \textbf{ until Loop} \\ x_{n+1} = \mathrm{prox}_{\gamma_n |E| \mathrm{TV}(\cdot, \xi_{n+1})}(x_n - \gamma_n L(\xi_{n+1}) \nabla F(x_n)) \end{cases}$$

Theorem [SBH'17] : Under mild assumptions, $x_n \longrightarrow_{n \to +\infty} x_\star$ where $x_\star \in \arg\min_{x \in \mathbb{R}^V} F(x) + \mathrm{TV}(x)$ a.s.

Proof:

- $\mathbb{E}\left(\mathrm{TV}(x, \xi)\right) = \frac{|E|}{L} \mathrm{TV}(x, G)$
- **A Generalized Stochastic Proximal Gradient Algorithm**

Illustration: Online Regularization

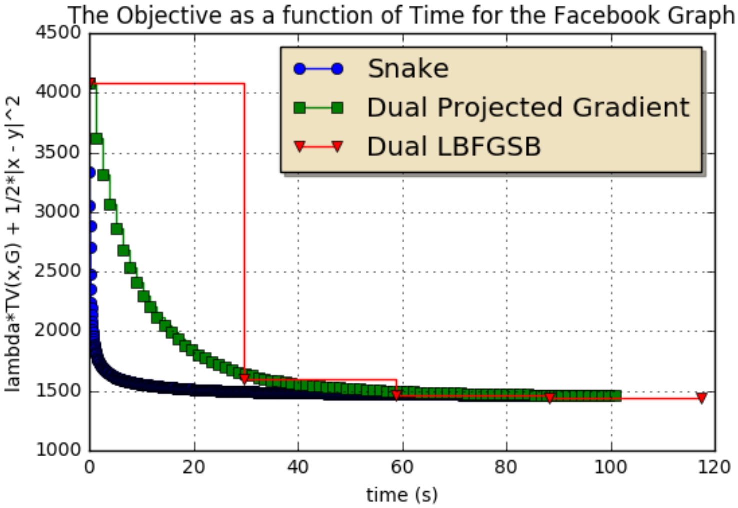

Figure 2: Snake: Trend Filtering over Facebook Graph Leskovec *et al.*'16

Structured Regularizations over Graphs

Generalization

$$\min_{x \in \mathbb{R}^V} F(x) + \sum_{\{i,j\} \in E} \phi_{i,j}(x(i), x(j))$$

with $\phi_{i,j}$ symmetric convex.

Example

- Weighted TV regularization
- Laplacian regularization :

$$\sum_{\{i,j\} \in E} \phi_{i,j}(x(i), x(j)) = \sum_{\{i,j\} \in E} (x(i) - x(j))^2$$

  Taut String $\leftarrow$ **DCT, IDCT**
- Weighted/Normalized Laplacian regularization

References

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